Higher order corrections to the Newtonian potential in the Randall-Sundrum model

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The general formalism for calculating the Newtonian potential in fine-tuned or critical Randall-Sundrum braneworlds is outlined. It is based on using the full tensor structure of the graviton propagator. This approach avoids the brane-bending effect arising from calculating the potential for a point source. For a single brane, this gives a clear understanding of the disputed overall factor 4/3 entering the correction. The result can be written on a compact form which is evaluated to high accuracy for both short and large distances.

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I. INTRODUCTION

During the recent years, there has been a lot of activity on theories with extra dimensions. In contrast to the Kaluza-Klein picture where the radius of the extra dimension is usually taken to be the Planck length, these new theories allow for millimeter-sized extra dimensions as first proposed by Arkani-Hamed *et al.* [1], or even an infinite extra dimension as proposed by Randall and Sundrum [2, 3]. These models were originally constructed to solve the hierarchy problem between the electroweak scale and the Planck scale and usually require Standard Model fields to be trapped on a 4-dimensional surface, called a 3-brane, to avoid conflict with experiments in particle physics where no signs of extra dimensions have been detected so far. Since gravity per definition can propagate through all dimensions, one important question is then whether gravity is still effectively localized on the brane, and Newton's law reproduced for distances larger than about a millimeter.

Starting with Randall and Sundrum [3], several other authors have calculated the Newtonian potential in the RS model, finding that $V(r) \sim \frac{1}{r}\left[1 + \Delta(r)\right]$, i.e. a dominant 4-dimensional term together with corrections. Although everyone agrees with the general form $\Delta \sim 1/\mu^2 r^2$ of the leading order correction (for large distances), there seems to be an ambiguity concerning the numerical coefficient. By looking at the scalar part of the graviton propagator, the result $\Delta = 1/(2\mu^2 r^2)$ is obtained [4, 5], whereas calculating the component h_{00} of the metric perturbation due to a matter source on the brane gives the result $\Delta = 2/(3\mu^2 r^2)$ [6, 7, 8, 9]. In the latter calculation, complications arise because the position of the brane can no longer be fixed in the fifth dimension when introducing a matter source. The relative factor 4/3 between the above results is then considered to be the net result of this brane-bending effect.

In this paper we calculate the Newtonian potential from the graviton propagator for a critical brane (with cosmological constant $\Lambda = 0$ and curvature $k = 0$), and show that the factor 4/3 is obtained by using the full tensor structure of the propagator instead of just the scalar part, without the need to include any brane-bending effects since we don't have a matter source [10]. This also gives the same general expression for the correction as obtained by Chung *et al.* [8], but derived in more detail here. Throughout this paper, we try to include most of the details in each calculation, hopefully making it possible to follow without much prior knowledge on the subject. The only parts left out are either straightforward or easily found elsewhere in the literature.

Other works which are closely related to this paper include [4] and [11], where an induced 4-dimensional curvature term (i.e. a term $\sim \int d^4x\sqrt{g}R$ in the action) modifies the Newtonian potential so that it behaves 4-dimensional even at very short distances, and [12], where n compact extra dimensions are included in addition to the large one. In [13] the conditions for localization of gravity are addressed more systematically, and in [14] it is shown that gravity is localized also for a non-critical brane, meaning that localization is a local property of the model, unaffected by the geometry of space far from the brane. The localization of other spin fields (spin 0, 1, $\frac{1}{2}$ and $\frac{3}{2}$) is studied in [15].

This paper is organized as follows: In section II we begin by solving Einstein's equations for the 5-dimensional RS model, thus obtaining its metric. In section III we make a perturbation to the metric and derive the wave equation the perturbation has to satisfy, starting with a completely general metric and then specializing to the RS case. The wave equation is then solved, giving us the graviton wavefunction. From the wavefunction we obtain the graviton propagator in section IV, and finally the Newtonian potential between two point masses on the brane, expressed as an integral over the mass m of the gravitons. In order to obtain the correct normalization for the wavefunction, and also the correct integration measure over the mass m, we use a regulator brane at a distance y_r from the physical brane, and take the limit

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$y_r \to \infty$ at the end of the calculation. We also expand the potential in powers of the distance r between the point masses, obtaining the leading order corrections $2/(3\mu^2 r^2)$ and $4/(3\pi\mu r)$ for large and short distances, respectively, along with higher order corrections.

II. THE RANDALL-SUNDRUM METRIC

We start out by assuming that the 5-dimensional metric has the form

$$ds^2 = n^2(t,y)dt^2 - a^2(t,y)\gamma_{ij}dx^i dx^j - dy^2, \tag{1}$$

where

$$\gamma_{ij} = \left(1 + \frac{k}{4}\delta_{lm}x^l x^m\right)^{-2}\delta_{ij} \tag{2}$$

is the spatial metric. The metric is thus parametrized by two scale factors $n(t,y)$ and $a(t,y)$, with y being the coordinate in the fifth dimension, and $y=0$ taken as the position of the brane. We use the standard convention $n(t,y=0)=1$, which means that t is the proper time on the brane. The energy momentum tensor is taken to be $T_{MN} = T^B_{MN} + T^b_{MN}$, where

$$T^B_{MN} = \mathrm{diag}(\rho_B n^2, p_B a^2\gamma_{ij}, p_B) \tag{3}$$

is the contribution from the bulk space (index B), and

$$T^b_{MN} = \mathrm{diag}(\rho_b n^2, p_b a^2\gamma_{ij}, 0)\delta(y) \tag{4}$$

is the contribution from the brane (index b). Finding the Einstein tensor and solving Einstein's equation is straightforward and has been done by many authors (see e.g. [16, 17, 18, 19]), so we simply state the results here. First, the Einstein tensor is given by

$$\begin{aligned}
E_{tt} &= 3\left[\frac{\dot a^2}{a^2} - n^2\left(\frac{a''}{a} + \frac{(a')^2}{a^2} - \frac{k}{a^2}\right)\right],\\
E_{ij} &= a^2\gamma_{ij}\left[\frac{(a')^2}{a^2} + \frac{2a'n'}{an} + \frac{2a''}{a} + \frac{n''}{n}\right.\\
&\qquad \left. -\frac{1}{n^2}\left(\frac{\dot a^2}{a^2} - \frac{2\dot a\dot n}{an} + \frac{2\ddot a}{a}\right) - \frac{k}{a^2}\right],\\
E_{ty} &= 3\left(\frac{n'\dot a}{na} - \frac{\dot a'}{a}\right),\\
E_{yy} &= 3\left[\frac{(a')^2}{a^2} + \frac{a'n'}{an} - \frac{1}{n^2}\left(\frac{\dot a^2}{a^2} - \frac{\dot a\dot n}{an} + \frac{\ddot a}{a}\right) - \frac{k}{a^2}\right],
\end{aligned} \tag{5}$$

where a dot denotes the derivative with respect to t, and a prime the derivative with respect to y. Assuming that the bulk space only contains a cosmological constant, i.e. $\rho_B = M^3\Lambda_B = -p_B$, Einstein's equation $E_{MN} = M^{-3}T_{MN}$ is solved by

$$a^2(t,y) = \begin{cases} a_0^2\left(1 - \frac{1}{3}M^{-3}\rho_b|y|\right) + (\dot a_0^2 + k)y^2, & \Lambda_B = 0,\\ \left[a_0^2 - \frac{3}{\Lambda_B}(\dot a_0^2 + k)\right]\cosh 2\mu y - \\ \quad \frac{\rho_b a_0^2}{6\mu M^3}\sinh 2\mu|y| + \frac{3}{\Lambda_B}(\dot a_0^2 + k), & \Lambda_B < 0,\\ \left[a_0^2 - \frac{3}{\Lambda_B}(\dot a_0^2 + k)\right]\cos 2\mu y - \\ \quad \frac{\rho_b a_0^2}{6\mu M^3}\sin 2\mu|y| + \frac{3}{\Lambda_B}(\dot a_0^2 + k), & \Lambda_B > 0, \end{cases} \tag{6}$$

and $n(t,y) = \dot a/\dot a_0$. Here M is the 5-dimensional Planck mass, $\mu = \sqrt{|\Lambda_B|/6}$, and $a_0 = a(t, y=0)$ denotes the scale factor on the brane. The latter satisfies an equation similar to Friedmann's first equation

$$\frac{\dot a_0^2}{a_0^2} = \frac{\rho_b^2}{36M^6} + \frac{\Lambda_B}{6} - \frac{k}{a_0^2} - \frac{U}{a_0^4}, \tag{7}$$

where U is a constant of integration, often called the dark radiation term for obvious reasons. In addition, we have the energy conservation equation

$$\dot\rho_b + 3\frac{\dot a_0}{a_0}(\rho_b + p_b), \tag{8}$$

just as in standard cosmology. Note, in particular, the absolute value $|y|$ in (6), which is the result of the boundary conditions

$$\frac{[a']}{a_0} = -\tfrac{1}{3}M^{-3}\rho_b, \qquad \frac{[n']}{n_0} = \tfrac{1}{3}M^{-3}(2\rho_b + 3p_b), \tag{9}$$

imposed by the delta function in T^b_{MN}. Here $[a']$ means the jump discontinuity in a' across the brane, i.e. $[a'] = a'(y=0^+) - a'(y=0^-) = 2a'(0^+)$. If we write the energy density ρ_b on the brane as $\rho_b = \lambda + \rho$ where λ is a constant, (7) takes the more recognizable form

$$\frac{\dot a_0^2}{a_0^2} = \frac{\rho}{3M_{\rm Pl}^2}\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\Lambda}{3} - \frac{k}{a_0^2} - \frac{U}{a_0^4}, \tag{10}$$

where we have identified the effective 4-dimensional Planck mass

$$M_{\rm Pl} = (M^{-6}\lambda/6)^{-1/2} = M^3\sqrt{6/\lambda}, \tag{11}$$

and the effective cosmological constant

$$\Lambda = \frac{1}{12}M^{-6}\lambda^2 + \frac{1}{2}\Lambda_B = \frac{1}{2}(M_{\rm Pl}^{-2}\lambda + \Lambda_B) \tag{12}$$

on the brane. The constant λ is often called the brane tension, and is assumed to be positive. Clearly, (10) reproduces standard 4-dimensional cosmology when U is "small" and λ "large". We will not discuss more precisely what this means other than saying that λ has to be (much) larger than the typical densities $\rho_{\rm nucl.}$ of matter and radiation during nucleosynthesis.

For the remainder of this paper, we will make the simplifying assumptions that $\rho_b = \lambda$ (the brane contains no components other than its tension) and that $U = 0$. Under these assumptions, the scale factor $a(t, y)$ can be factorized as $a(t, y) = a_0(t)A(y)$, where $A(y)$ follows from (6):

$$A(y) = \begin{cases} 1 - \dfrac{\lambda}{6M^3}|y|, & \Lambda_B = 0, \\ \cosh \mu y - \dfrac{\lambda}{6\mu M^3} \sinh \mu|y|, & \Lambda_B < 0, \\ \cos \mu y - \dfrac{\lambda}{6\mu M^3} \sin \mu|y|, & \Lambda_B > 0. \end{cases} \tag{13}$$

It also follows that $n(t, y) = A(y)$, and the metric is simplified to

$$ds^2 = A^2(y) g_{\mu\nu}(x) dx^\mu dx^\nu - dy^2, \tag{14}$$

where $g_{\mu\nu}(x) = dt^2 - a_0^2(t)\gamma_{ij} dx^i dx^j$ is the standard Friedmann-Robertson-Walker (FRW) metric on the brane.

Our main focus in this paper will be the case of a vanishing effective cosmological constant, $\Lambda = 0$, which is often called a critical brane. We then get $\Lambda_B = -\frac{1}{6}M^{-6}\lambda^2 < 0$, meaning that the bulk space is AdS_5, and $A(y) = e^{-\mu|y|}$. If we also assume a spatially flat universe, $k = 0$, the metric becomes

$$ds^2 = e^{-2\mu|y|}\eta_{\mu\nu} dx^\mu dx^\nu - dy^2. \tag{15}$$

This is the case originally studied in [2, 3]. One should note that $\Lambda_B < 0$ is in fact the only possibility compatible with observations. Since the observed value of Λ is very small, the requirement that $\lambda \gg \rho_{\rm nucl.}$ obviously gives $M_{\rm Pl}^{-2}\lambda \gg |\Lambda|$. From (12) we therefore get $\Lambda_B = 2\Lambda - M_{\rm Pl}^{-2}\lambda \approx -M_{\rm Pl}^{-2}\lambda < 0$.

III. LOCALIZATION OF GRAVITY

The main goal of this paper is to calculate the correction to Newton's law of gravitation due to the presence of the fifth dimension. Since the Newtonian potential is essentially the low energy limit of the graviton propagator, we must find the graviton wavefunction and in particular study its localization on the brane. The graviton is described by the traceless transverse component h_{ij} of the spatial metric perturbation. The question of finding the linearized wave equation satisfied by h_{ij} has been addressed by many authors [17, 20, 21], but since this is an integral part of the calculation, we include the details here.

A. General linearized gravity

To begin with, we assume nothing about the background metric $g_{\mu\nu}$, and consider the general perturbed metric

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \tag{16}$$

where also the number of dimensions is arbitrary. In the following, we will only keep terms to the first order in h and its derivatives. The inverse metric is therefore

$$\hat{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}, \tag{17}$$

and the Christoffel symbols

$$\begin{aligned}\hat{\Gamma}^\mu{}_{\alpha\beta} &= \Gamma^\mu{}_{\alpha\beta} + \tfrac{1}{2} g^{\mu\nu}\left(h_{\nu\alpha;\beta} + h_{\nu\beta;\alpha} - h_{\alpha\beta;\nu}\right) \\ &\equiv \Gamma^\mu{}_{\alpha\beta} + S^\mu{}_{\alpha\beta} \,.\end{aligned} \tag{18}$$

From this, we see that the perturbation $S^\mu{}_{\alpha\beta}$ to the Christoffel symbols is in fact a tensor. (The symbol ; means covariant derivative with respect to the background metric $g_{\mu\nu}$.) Next, we find the Riemann tensor

$$\begin{aligned}\hat{R}^\mu{}_{\nu\alpha\beta} &= R^\mu{}_{\nu\alpha\beta} + S^\mu{}_{\nu\beta,\alpha} - S^\mu{}_{\nu\alpha,\beta} \\ &\quad + \Gamma^\rho{}_{\nu\beta} S^\mu{}_{\rho\alpha} + \Gamma^\mu{}_{\rho\alpha} S^\rho{}_{\nu\beta} - \Gamma^\rho{}_{\nu\alpha} S^\mu{}_{\rho\beta} - \Gamma^\mu{}_{\rho\beta} S^\rho{}_{\nu\alpha} \\ &= R^\mu{}_{\nu\alpha\beta} + S^\mu{}_{\nu\beta;\alpha} - S^\mu{}_{\nu\alpha;\beta} \,,\end{aligned} \tag{19}$$

and thus the Ricci tensor

$$\begin{aligned}\hat{R}_{\mu\nu} &= \hat{R}^\alpha{}_{\mu\alpha\nu} = R_{\mu\nu} + S^\alpha{}_{\mu\nu;\alpha} - S^\alpha{}_{\mu\alpha;\nu} \\ &= R_{\mu\nu} + \tfrac{1}{2}\left(h_{\alpha\mu;\nu}{}^\alpha + h_{\alpha\nu;\mu}{}^\alpha - h_{\mu\nu;\alpha}{}^\alpha - h_{;\mu\nu}\right),\end{aligned} \tag{20}$$

where the last expression is obtained after a short and straightforward calculation. Finally, we obtain the curvature scalar

$$\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu} = R - h^{\mu\nu}R_{\mu\nu} + h^{\mu\nu}{}_{;\mu\nu} - \Box h \,, \tag{21}$$

and the Einstein tensor

$$\begin{aligned}\hat{E}_{\mu\nu} &= \hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R}\hat{g}_{\mu\nu} \\ &= E_{\mu\nu} + \tfrac{1}{2}\left(h_{\alpha\mu;\nu}{}^\alpha + h_{\alpha\nu;\mu}{}^\alpha - h_{\mu\nu;\alpha}{}^\alpha - h_{;\mu\nu}\right) \\ &\quad - \tfrac{1}{2}Rh_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}\left(h^{\alpha\beta}R_{\alpha\beta} + \Box h - h^{\alpha\beta}{}_{;\alpha\beta}\right).\end{aligned} \tag{22}$$

This can be somewhat simplified by using the trace-reversed components $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$, yielding

$$\begin{aligned}\delta E_{\mu\nu} &\equiv \hat{E}_{\mu\nu} - E_{\mu\nu} \\ &= \tfrac{1}{2}(\bar{h}_{\alpha\mu;\nu}{}^\alpha + \bar{h}_{\alpha\nu;\mu}{}^\alpha - \nabla_\alpha^2\bar{h}_{\mu\nu} - R\bar{h}_{\mu\nu} \\ &\quad + g_{\mu\nu}\bar{h}^{\alpha\beta}R_{\alpha\beta} - g_{\mu\nu}\bar{h}^{\alpha\beta}{}_{;\alpha\beta}) \,.\end{aligned} \tag{23}$$

Throughout this paper $\Box$ always means the scalar d'Alembertian, i.e. $\Box \equiv (1/\sqrt{g})\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$, whereas $\nabla_\alpha^2 \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ means the covariant Laplacian which acts differently on tensors of different rank. It is therefore only for a scalar field ϕ that $\nabla_\alpha^2\phi = \Box\phi$.

The invarianse of (23) under an arbitrary coordinate transformation can be used to simplify the expression. We choose to work in the gauge where $\bar{h} = 0$ and $\bar{h}^{\mu\nu}{}_{;\nu} =$

0. In order to utilize the latter condition in (23), we use the general tensor identity

$$T_{\alpha\beta;\mu\nu} - T_{\alpha\beta;\nu\mu} = T_{\alpha\sigma} R^{\sigma}{}_{\beta\mu\nu} + T_{\sigma\beta} R^{\sigma}{}_{\alpha\mu\nu}\,, \tag{24}$$

which can be proven by "brute force" by writing out the covariant derivatives explicitely. From this it follows that

$$\bar{h}_{\alpha\mu;\nu}{}^{\alpha} = \bar{h}^{\alpha}{}_{\mu;\alpha\nu} + \bar{h}^{\alpha}{}_{\sigma} R^{\sigma}{}_{\mu\nu\alpha} + \bar{h}^{\sigma}{}_{\mu} R_{\sigma\nu}\,, \tag{25}$$

and (23) is then reduced to

$$\begin{aligned}\delta E_{\mu\nu} = \bar{h}^{\alpha}{}_{\sigma} R^{\sigma}{}_{\mu\nu\alpha} + \tfrac{1}{2}\big(\bar{h}^{\sigma}{}_{\mu} R_{\sigma\nu} + \bar{h}^{\sigma}{}_{\nu} R_{\sigma\mu} \\ - \nabla^2_{\alpha} \bar{h}_{\mu\nu} - R\bar{h}_{\mu\nu} + g_{\mu\nu} \bar{h}^{\alpha\beta} R_{\alpha\beta}\big)\,.\end{aligned} \tag{26}$$

Taking the trace, the first three terms are seen to cancel, yielding

$$\delta E^{\mu}{}_{\mu} = \frac{1}{2}\big(-\Box\bar{h} - R\bar{h} + D\bar{h}^{\alpha\beta} R_{\alpha\beta}\big) = \frac{D}{2}\bar{h}^{\alpha\beta} R_{\alpha\beta}, \tag{27}$$

where D is the number of spacetime dimensions, and we have used the traceless condition $\bar{h} = 0$. From now on, we can also ignore the bar, since $\bar{h}_{\mu\nu} = h_{\mu\nu}$.

Einstein's equation for the perturbed metric can be written $\hat{E}_{\mu\nu} = M_D^{2-D}\hat{T}_{\mu\nu}$, where M_D is the D-dimensional Planck mass, and $\hat{T}_{\mu\nu}$ the perturbed energy momentum tensor. Assuming that both energy density, pressure and "D-velocity" u^{μ} is unperturbed, i.e. $\delta\rho = \delta p = 0$ and $\delta u^{\mu} = 0$, we get

$$\begin{aligned}\hat{T}_{\mu\nu} &= (\rho + p) u_{\mu} u_{\nu} - p\hat{g}_{\mu\nu} \\ &= (\rho + p) u_{\mu} u_{\nu} - p(g_{\mu\nu} + h_{\mu\nu})\,,\end{aligned} \tag{28}$$

meaning that $\delta T_{\mu\nu} = -p h_{\mu\nu}$. The background metric is determined from the unperturbed energy momentum tensor, i.e. $E_{\mu\nu} = M_D^{2-D} T_{\mu\nu}$, and the equation for the perturbed Einstein tensor is therefore

$$\delta E_{\mu\nu} = M_D^{2-D}\,\delta T_{\mu\nu} = -M_D^{2-D} p h_{\mu\nu}\,. \tag{29}$$

From (27) we then get

$$\delta E^{\mu}{}_{\mu} = \frac{D}{2} h^{\alpha\beta} R_{\alpha\beta} = -M_D^{2-D} p h^{\mu}{}_{\mu} = 0, \tag{30}$$

since h is traceless. When inserted into (26), this gives

$$\begin{aligned}2h^{\alpha}{}_{\sigma} R^{\sigma}{}_{\mu\nu\alpha} + h^{\sigma}{}_{\mu} R_{\sigma\nu} + h^{\sigma}{}_{\nu} R_{\sigma\mu} \\ - \nabla^2_{\alpha} h_{\mu\nu} - R h_{\mu\nu} + 2M_D^{2-D} p h_{\mu\nu} = 0\,.\end{aligned} \tag{31}$$

The first four terms are often called the *de Rham-Lichnerowicz operator* $\triangle$ applied to $h_{\mu\nu}$:

$$\triangle h_{\mu\nu} \equiv 2h^{\alpha}{}_{\sigma} R^{\sigma}{}_{\mu\nu\alpha} + h^{\sigma}{}_{\mu} R_{\sigma\nu} + h^{\sigma}{}_{\nu} R_{\sigma\mu} - \nabla^2_{\alpha} h_{\mu\nu}\,. \tag{32}$$

The general equation (31) can be applied to any geometry we are interested in for tensor fluctuations.

B. Linearized gravity in the RS model

We will now apply (31) to the RS background metric (1). In addition to $h^{MN}{}_{;N} = 0$ and $h^{M}{}_{M} = 0$, we choose the gauge where $h_{\mu 4} = h_{44} = 0$. Having only one brane, this was shown possible in a more detailed discussion about gauge transformations by Boos *et al.* [22]. The condition that $h_{44} = 0$ corresponds to having no massless scalar field (radion) in the problem. (With two branes such a gauge choice is not possible, and the radion will contribute in that case.)

We now have a total of 10 gauge conditions, reducing the number of independent components of the symmetric tensor h_{MN} from 15 to 5. These components must then be separated into scalar, vector and tensor parts with respect to the spatial metric γ_{ij}. Here we choose to work only with the tensor part which describes pure gravitational waves, meaning that we set $h_{\mu 0} = 0$. The remaining part h_{ij} therefore have two independent degrees of freedom. If we also factorize out a^2 by letting $h_{ij} \to a^2 h_{ij}$, the full perturbed metric can be written

$$ds^2 = n^2(t,y)dt^2 - a^2(t,y)(\gamma_{ij} + h_{ij})dx^i dx^j - dy^2. \tag{33}$$

The indices of h_{ij} are now raised and lowered by γ_{ij}, and the gauge conditions $h^{MN}{}_{;N} = 0$ and $h^{M}{}_{M} = 0$ are reduced to $\nabla_j h^{ij} = 0$ and $h^i{}_i = \gamma^{ij} h_{ij} = 0$, where ∇_j is the spatial covariant derivative. Thus h_{ij} may be regarded effectively as a 3-dimensional tensor, with t and y as additional parameters. The different terms in (31) are then easily found using Mathematica [23], with the result

$$2h^K_L R^L_{MNK} + h^K_M R_{KN} + h^K_N R_{KM} = \left[-\frac{6k}{a^2} + \frac{6(a')^2}{a^2} + \frac{2a'n'}{an} + \frac{2a''}{a} - \frac{6\dot{a}^2}{a^2n^2} + \frac{2\dot{a}\dot{n}}{an^3} - \frac{2\ddot{a}}{an^2} \right] h_{MN} \,,$$
$$R = R^M_M = -\frac{6k}{a^2} + \frac{6(a')^2}{a^2} + \frac{6a'n'}{an} + \frac{6a''}{a} + \frac{2n''}{n} - \frac{6\dot{a}^2}{a^2n^2} + \frac{6\dot{a}\dot{n}}{an^3} - \frac{6\ddot{a}}{an^2} \,,$$
$$2M^{-3}p = -\frac{2k}{a^2} + \frac{2(a')^2}{a^2} + \frac{4a'n'}{an} + \frac{4a''}{a} + \frac{2n''}{n} - \frac{2\dot{a}^2}{a^2n^2} + \frac{4\dot{a}\dot{n}}{an^3} - \frac{4\ddot{a}}{an^2} \,. \qquad (34)$$

The last equation also follows directly from the Einstein tensor (5), since $E_{ij} = M^{-3}T_{ij} = M^{-3}pa^2\gamma_{ij}$, with $p = p_B + p_b\delta(y)$. Adding everything together, we see that most of the terms cancel, and we are left with

$$\frac{1}{a^2}\nabla^2_K(a^2h_{ij}) = \left[-\frac{2k}{a^2} + \frac{2(a')^2}{a^2} - \frac{2\dot{a}^2}{a^2n^2} \right] h_{ij} \,. \qquad (35)$$

A tedious but straightforward calculation shows that the 5-dimensional Laplacian can be expanded as

$$\frac{1}{a^2}\nabla^2_K(a^2h_{ij}) = \left[\frac{1}{n^2}\partial_0^2 + \left(\frac{3\dot{a}}{an^2} - \frac{\dot{n}}{n^3} \right)\partial_0 - \frac{1}{a^2}\gamma^{kl}\nabla_k\nabla_l - \partial_y^2 - \left(\frac{3a'}{a} + \frac{n'}{n} \right)\partial_y + \frac{2(a')^2}{a^2} - \frac{2\dot{a}^2}{a^2n^2} \right] h_{ij} \,, \qquad (36)$$

with the result

$$\left[\frac{1}{n^2}\partial_0^2 + \left(\frac{3\dot{a}}{an^2} - \frac{\dot{n}}{n^3} \right)\partial_0 - \frac{1}{a^2}\gamma^{kl}\nabla_k\nabla_l - \partial_y^2 - \left(\frac{3a'}{a} + \frac{n'}{n} \right)\partial_y + \frac{2k}{a^2} \right] h_{ij} = 0 \,. \qquad (37)$$

(Removing the y-dependence and setting $n = 1$, we recover the result for the 4-dimensional FRW metric as obtained by Ford and Parker [21].) We should compare (37) to the wave equation of a scalar field with mass m_ϕ in five dimensions:

$$(\Box + m_\phi^2)\phi = \left[\frac{1}{n^2}\partial_0^2 + \left(\frac{3\dot{a}}{an^2} - \frac{\dot{n}}{n^3} \right)\partial_0 - \frac{1}{a^2}\gamma^{kl}\nabla_k\nabla_l - \partial_y^2 - \left(\frac{3a'}{a} + \frac{n'}{n} \right)\partial_y + m_\phi^2 \right] \phi = 0 \,. \qquad (38)$$

The only difference between the two expressions lies in the tensor structure of the term $\gamma^{kl}\nabla_k\nabla_l h_{ij}$. The graviton field has also acquired an effective mass $m_k^2 = 2k/a^2$, which at the present epoch is of the order 10^{-33} eV (at most), and therefore totally negligible at short (i.e. non-cosmological) distances. For a spatially flat universe ($k = 0$), (37) is reduced to $\Box h_{ij} = 0$. The components of h_{ij} independently satisfy the Klein-Gordon equation in this case, and the graviton field is thus equivalent to a set of two independent scalar fields.

From (37) we see that the $\mathbf{x}$-dependence of h_{ij} can be separated from the t- and y-dependence. Writing $h_{ij} = \psi(t,y)G_{ij}(\mathbf{x})$, (37) is solved by $\gamma^{kl}\nabla_k\nabla_l G_{ij} = -\sigma^2 G_{ij}$ and

$$\left[\frac{1}{n^2}\partial_0^2 + \left(\frac{3\dot{a}}{an^2} - \frac{\dot{n}}{n^3} \right)\partial_0 - \partial_y^2 - \left(\frac{3a'}{a} + \frac{n'}{n} \right)\partial_y + \frac{2k + \sigma^2}{a^2} \right] \psi(t,y) = 0 \,. \qquad (39)$$

The eigenvalues σ^2 of the 3-dimensional Laplacian can be found in [21], but we will not need them here. Instead, since we are considering the factorized metric (14), it is more convenient to separate the y-dependence only.

Substituting $a(t,y) = a_0(t)A(y)$ and $n(t,y) = A(y)$ into (37), it simplifies to

$$\left[\partial_y^2 + \frac{4A'}{A}\partial_y - \frac{1}{A^2} \times \left(\partial_0^2 + \frac{3\dot{a}_0}{a_0}\partial_0 - \frac{1}{a_0^2}\gamma^{kl}\nabla_k\nabla_l + \frac{2k}{a_0^2} \right) \right] h_{ij} = 0 \,, \qquad (40)$$

with the solution $h_{ij}(x,y) = G_{ij}(x)\Phi(y)$, where

$$\left(\partial_0^2 + \frac{3\dot{a}_0}{a_0}\partial_0 - \frac{1}{a_0^2}\gamma^{kl}\nabla_k\nabla_l + \frac{2k}{a_0^2} + m^2 \right) G_{ij}(x) = 0, \qquad (41)$$
$$\Phi''(y) + \frac{4A'}{A}\Phi'(y) + \frac{m^2}{A^2}\Phi(y) = 0, \qquad (42)$$

and m is the eigenvalue following from the separation of variables. This is the same result as obtained by Brevik *et al.* [18]. Explicit solutions to (41) can be found by separating the t-dependence, but we will not need them in the following. The mass spectrum will be determined from (42).

One should also note that the result (42) can be obtained without demanding that $h_{\mu 0} = 0$ when the 5-dimensional metric has the simple form (14). Instead, the perturbed metric can then be written

$$ds^2 = A^2(y)\left(g_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu - dy^2 \,, \qquad (43)$$

with the gauge conditions $h^{\mu\nu}{}_{;\nu} = 0$ and $h = g^{\mu\nu}h_{\mu\nu} = 0$, where the indices of $h_{\mu\nu}$ are raised and lowered by $g_{\mu\nu}$. Thus $h_{\mu\nu}$ is effectively a 4-dimensional tensor, with five independent degrees of freedom. Going through the same

calculations as above, starting from (31), we then get the result

$$\left[\partial_y^2 + \frac{4A'}{A}\partial_y - \frac{1}{A^2}\left(\nabla_\alpha^2 + \frac{2\Lambda}{3}\right)\right] h_{\mu\nu} = 0\,, \qquad (44)$$

where ∇_α^2 is the 4-dimensional Laplacian. The apparent difference in the effective mass term compared to (40) is simply because now we have not factorized out $a_0^2(t)$ from the perturbation. Writing $h_{\mu\nu} = a_0^2(t)\tilde{h}_{\mu\nu}$ and using Friedmann's equation (10), $\tilde{h}_{\mu\nu}$ can be seen to satisfy (40), provided we also set $\tilde{h}_{\mu 0} = 0$ in order to expand ∇_α^2. Equation (44) is solved by setting $h_{\mu\nu}(x,y) = G_{\mu\nu}(x)\Phi(y)$, where $\Phi(y)$ satisfies (42), and

$$\left(\nabla_\alpha^2 + \frac{2\Lambda}{3} + m^2\right) G_{\mu\nu}(x) = 0\,. \qquad (45)$$

Thus, m can be regarded as the effective mass of the 5-dimensional graviton field as observed in four dimensions, with a small correction from the cosmological constant. When $\Lambda > 0$, even the $m = 0$ state is massive. This appears to create a serious problem, since massive gravitons (in 4 dimensions) have five polarization states whereas massless gravitons only have two polarization states, and all physical results should be continuous in the limit $\Lambda \to 0$. This problem and its solution has been discussed by Karch and Randall [14].

C. The graviton wavefunction

Next, we proceed to solve (42). Since this has already been done [3, 5, 18], we will only sketch the derivation. First, we simplify the equation by making the substitution $y \to z(y)$, where

$$\frac{\partial y}{\partial z} = A(y), \qquad (46)$$

and by writing $\Phi(y)$ as

$$\Phi(y) = A^{-3/2}u(z). \qquad (47)$$

This leads to the Schrödinger like equation for $u(z)$

$$\left[-\partial_z^2 + V(z)\right] u(z) = m^2 u(z), \qquad (48)$$

where the effective potential $V(z)$ is given by

$$V(z) = \frac{9}{4}(A')^2 + \frac{3}{2}AA''. \qquad (49)$$

Notice that the line element (14) takes the form

$$ds^2 = A^2(z)\left(g_{\mu\nu}dx^\mu dx^\nu - dz^2\right) \qquad (50)$$

when expressed using the coordinate z. Therefore z is often called the conformal coordinate.

Focusing on the critical case $\Lambda = 0$ where $A(y) = e^{-\mu|y|}$, we get the result

$$\begin{aligned} z &= \mathrm{sgn}(y)\frac{1}{\mu}\left(e^{\mu|y|} - 1\right), \\ V(z) &= \frac{15\mu^2}{4\left(1+\mu|z|\right)^2} - 3\mu\delta(z)\,, \end{aligned} \qquad (51)$$

where the delta function originates from the double derivative of $|y|$, and we have used the general relation $\delta(z - z') = A(y')\delta(y - y')$ together with $A(0) = 1$. The potential for the case $\Lambda > 0$ can be found in e.g. [18, 24], and in [14] the case $\Lambda < 0$ is also included. The delta function in the result means that $u(z)$ must depend on the absolute value $|z|$ (just as the delta function in T_{MN} resulted in $a(t,y)$ depending on $|y|$). Matching the delta function in $V(z)$ with the delta function from $u''(z)$, we get the boundary condition

$$2u'(0) + 3\mu u(0) = 0\,, \qquad (52)$$

and the equation for $z > 0$

$$u''(z) + \left[m^2 - \frac{15\mu^2}{4(1+\mu z)^2}\right] u(z) = 0\,. \qquad (53)$$

The general solution to (53) is easily shown to be $u(z) = \sqrt{1+\mu z}\left\{AJ_2[\frac{m}{\mu}(1+\mu z)] + BY_2[\frac{m}{\mu}(1+\mu z)]\right\}$ for arbitrary constants A and B, where $J_2(x)$ and $Y_2(x)$ are the Bessel functions of order two of the first and second kind, respectively. In order to satisfy the boundary condition (52), we must choose the linear combination

$$\begin{aligned} u(z) &= N_m\sqrt{1+\mu|z|} \\ &\times\left\{Y_2[\tfrac{m}{\mu}(1+\mu|z|)] - \frac{Y_1(\frac{m}{\mu})}{J_1(\frac{m}{\mu})}J_2[\tfrac{m}{\mu}(1+\mu|z|)]\right\}, \end{aligned} \qquad (54)$$

where N_m is a normalization constant (to be found later). This expression is valid for all m, so we have a continuous mass spectrum for $m > 0$. The solution for $m = 0$ can be found either directly from (53) or by taking the limit $m \to 0$ in (54), and is given by

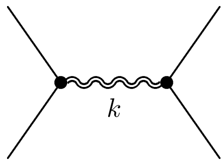
$$u_0(z) = N_0(1+\mu|z|)^{-3/2}. \qquad (55)$$

Thus gravity is localized on the brane, since the massless, bound state $u_0(z)$ can be normalized, and has a sharp peak at $z = 0$. As we will see later, the massive states (54) give the deviation from Newton's law. Solutions for the case $\Lambda > 0$ are given in [5, 18, 24], and for $\Lambda < 0$ in [14].

IV. CORRECTIONS TO NEWTON'S LAW

Having found the graviton wavefunction, we will use this result to calculate the static gravitational potential between two point masses both located on the brane. As

is well known from field theory, the (Fourier transformed) potential is obtained by looking at a scattering diagram where two particles interact through the exchange of a virtual graviton, in the limit where the energy of the graviton goes to zero:

$$V(\mathbf{k}) = \lim_{k^0\to 0}$$  $$. \qquad (56)$$

We therefore need to know the 5-dimensional graviton propagator, as well as the interaction between a graviton and a massive particle on the brane. The propagator must in principle be found by inverting the quadratic terms in the effective graviton Lagrangian [22]. However, since we have expressed the 5-dimensional graviton wavefunction directly in terms of the 4-dimensional ones through $h_{\mu\nu}(x,y) = \sum_m G^m_{\mu\nu}(x)\Phi(m,y)$, we know that the 5-dimensional propagator must be similarly expressed by the 4-dimensional propagator, and can therefore immediately write down the answer:

$$\begin{aligned} D^{(5)}_{\mu\nu\alpha\beta}(x,y;x',y') &\equiv \langle 0|T\hat{h}_{\mu\nu}(x,y)\hat{h}_{\alpha\beta}(x',y')|0\rangle \\ &= \sum_m \Phi(m,y)\Phi^*(m,y')D^{(4,m)}_{\mu\nu\alpha\beta}(x,x'), \end{aligned} \qquad (57)$$

where $D^{(4,m)}_{\mu\nu\alpha\beta}(x,x')$ is the propagator of a 4-dimensional spin-2 particle with mass m. Taking $y = y' = 0$ and using $\Phi(m,y) = A(y)^{-3/2}u(m,z)$, we get

$$D^{(5)}_{\mu\nu\alpha\beta}(x,0;x',0) = \sum_m |u(m,0)|^2 D^{(4,m)}_{\mu\nu\alpha\beta}(x,x'). \qquad (58)$$

Everything is now happening in just four dimensions. The only trace left of the motion of the graviton through the fifth dimension is that we have a tower of massive 4-dimensional gravitons with a non-trivial "weight" $|u(m,0)|^2$. More importantly, the propagator $D^{(4,m)}_{\mu\nu\alpha\beta}(x,x')$ is completely described in the 4-dimensional FRW space with metric $g_{\mu\nu}$, where standard results can be applied directly without worrying about the fifth dimension.

Focusing again on the critical case $k = 0$ and $\Lambda = 0$, the 4-dimensional space is flat ($g_{\mu\nu} = \eta_{\mu\nu}$), with the result

$$D^{(4,m)}_{\mu\nu\alpha\beta}(x,x') = \int \frac{d^4k}{(2\pi)^4}\frac{P^{(m)}_{\mu\nu\alpha\beta}(k)}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-x')}, \qquad (59)$$

where the polarization tensor can be chosen as (see e.g. [25] for more details)

$$P^{(m=0)}_{\mu\nu\alpha\beta}(k) = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right), \qquad (60)$$

$$\begin{aligned} P^{(m>0)}_{\mu\nu\alpha\beta}(k) &= \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right) \\ &\quad -\frac{1}{2m^2}\left(\eta_{\mu\alpha}k_\nu k_\beta + \eta_{\mu\beta}k_\nu k_\alpha + \eta_{\nu\alpha}k_\mu k_\beta + \eta_{\nu\beta}k_\mu k_\alpha\right) \\ &\quad +\frac{1}{6}\left(\eta_{\mu\nu} + \frac{2}{m^2}k_\mu k_\nu\right)\left(\eta_{\alpha\beta} + \frac{2}{m^2}k_\alpha k_\beta\right). \end{aligned} \qquad (61)$$

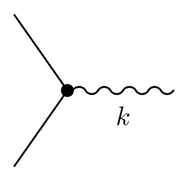
Next, we consider the interaction. For a Lagrangian $\mathcal{L}$ in D dimensions the action is $S = \int d^Dx\sqrt{g}\mathcal{L}$. The energy momentum tensor T_{MN} corresponding to the Lagrangian $\mathcal{L}$ is defined by varying the action with respect to the metric:

$$\delta S \equiv \int d^Dx\sqrt{g}\frac{1}{2}T_{MN}\delta g^{MN}. \qquad (62)$$

Since $\delta g^{MN} = -h^{MN}$ to the lowest order in h, we therefore get the interaction Lagrangian

$$\mathcal{L}_{\rm int} = -\frac{1}{2}T_{MN}h^{MN}. \qquad (63)$$

So far, we have considered the metric perturbation h_{MN} to be dimensionless, since g_{MN} is dimensionless. But when treating it as a field of particles (i.e. gravitons), this can no longer be the case if we want to have a canonical kinetic term in the Lagrangian. Instead, h_{MN} has to have a dimension $\dim[h] = (D-2)/2$ in D spacetime dimensions. Since h_{MN} is a 5-dimensional field, we must therefore have $\dim[h] = 3/2$. From $h_{\mu\nu}(x,y) = G_{\mu\nu}(x)\Phi(y)$, we then get $\dim[\Phi] = \dim[u] = 1/2$, since $G_{\mu\nu}$ is a 4-dimensional field and should therefore have dimension 1. This is also required in order to make equations (57-61) consistent. Again, since $h_{\mu\nu}$ is 5-dimensional, we should use the 5-dimensional Planck mass M and let $h^{\mu\nu} \to M^{-3/2}h^{\mu\nu}$ in (63). Finally, using $M^3 = \mu M^2_{\rm Pl}$ which follows from (11) and (12), we get the result



$$= \frac{1}{\sqrt{\mu}M_{\rm Pl}}T^{\mu\nu}(k). \qquad (64)$$

Before proceeding, lets give a little thought to the assumption $\Lambda = 0$ and $k = 0$. For nonzero Λ, we should expect corrections to Newton's law of the form $(|\Lambda|r^2)^n$ for some positive constant n. Because of the very small observed value of Λ, this correction will only be noticable at very large distances. But the Newtonian potential is only a meaningful quantity at short (i.e. non-cosmological) distances, so for our purpose the approximation $\Lambda \approx 0$ should be perfectly valid. The same is also true for a potentially non-vanishing k. From a pure mathematical point of view, however, it would be interesting to consider an arbitrary value of Λ. But in that case, one has to use the full deSitter propagator instead of (59).

Another way of looking at this problem, is to note that the effect of the fifth dimension and the effect of Λ and k takes place at completely different scales. The effect of

the fifth dimension dominates at very short distances (as we will see), where we can set $\Lambda = 0$ and $k = 0$, whereas the effect of Λ and k dominates at very large distances, where we can ignore the fifth dimension altogether and simply use 4-dimensional FRW space. Trying to include both effects in the same calculation will only make things unnecessarily complicated.

A. The Newtonian potential

Combining the above results (59) and (64) for the propagator and the vertex and summing over m, we finally obtain the gravitational potential

$$V(\mathbf{k}) = \sum_m |u(m,0)|^2 \frac{1}{\mu M_{\mathrm{Pl}}^2} \frac{T_1^{\mu\nu} P^{(m)}_{\mu\nu\alpha\beta} T_2^{\alpha\beta}}{|k^2 - m^2|}\Bigg|_{k^0\to 0} . \quad (65)$$

Taking both particles to be point particles at rest with masses m_1 and m_2, the energy momentum tensors can be written $T_1^{\mu\nu}(\mathbf{x}) = m_1\delta(\mathbf{x})u^\mu u^\nu \Rightarrow T_1^{\mu\nu}(\mathbf{k}) = m_1 u^\mu u^\nu = m_1\delta_0^\mu\delta_0^\nu$, and likewise for $T_2^{\alpha\beta}$. We therefore see that only the 0000-component of the polarization tensor contributes to the potential:

$$V(\mathbf{k}) = \frac{8\pi G m_1 m_2}{\mu} \sum_m |u(m,0)|^2 \frac{P^{(m)}_{0000}(\mathbf{k})}{\mathbf{k}^2 + m^2} . \quad (66)$$

(Here we have written $M_{\mathrm{Pl}}^{-2} = 8\pi G$, where G is the gravitational constant.) Now, since $k_0 = 0$, the polarization tensor is easily found from (60) and (61):

$$P^{(m)}_{0000}(\mathbf{k}) = \begin{cases} \frac{1}{2} , & m = 0 , \\ \frac{1}{2} + \frac{1}{6} = \frac{2}{3} , & m > 0 , \end{cases} \quad (67)$$

with the result

$$V(\mathbf{k}) = \frac{8\pi G m_1 m_2}{\mu} \left\{ \frac{1}{2}\frac{|u(0,0)|^2}{\mathbf{k}^2} + \frac{2}{3}\sum_{m>0} \frac{|u(m,0)|^2}{\mathbf{k}^2 + m^2} \right\} , \quad (68)$$

or

$$V(r) = \frac{G m_1 m_2}{\mu r} \left\{ |u(0,0)|^2 + \frac{4}{3}\sum_{m>0} |u(m,0)|^2 e^{-mr} \right\} . \quad (69)$$

Thus, we see the importance of using the full tensor structure of the propagator, instead of just the scalar part which is commonly used. Without the tensor part we would miss the relative factor 4/3 between the massless and the massive modes.

In order to proceed further, we need the normalization constants N_m and N_0 in (54) and (55), which in principle are found by requiring

$$\int_{-\infty}^{\infty} |u(0,z)|^2 dz = 1 ,$$
$$\int_{-\infty}^{\infty} u(m,z)^* u(m',z) dz = \delta(m - m') . \quad (70)$$

However, the second integral as it stands is divergent for all m and m', not just for $m = m'$, and therefore has to be regulated in some way. As mentioned in [3], this can be done by introducing a regulator brane at some large but finite position z_r, and then taking the limit $z_r \to \infty$ at the end. Since this point seems to be taken very lightly in the literature, we will go through the calculations in detail here.

B. Using a regulator brane

With a second brane located at $y = y_r$, we still assume that the points y and $-y$ are equal, which means that the fifth dimension is now compactified on an orbifold S^1/Z_2. The two branes thus represent the endpoints of the fifth dimension. Einstein's equation in the bulk space and the boundary conditions on the physical brane are unchanged by the presence of the regulator brane, and the only way it manifests itself is as an additional set of boundary conditions. (Several works have been done in the two brane scenario where both branes are considered physical, see e.g. [19]. In this paper we are only using the second brane as a regulator, therefore ignoring its cosmological implications.) From the energy momentum tensor $T^r_{MN} = \mathrm{diag}(\rho_r n^2, p_r a^2\gamma_{ij}, 0)\delta(y - y_r)$ on the regulator brane, we get

$$\frac{[a']}{a}\bigg|_{y=y_r} = -\tfrac{1}{3}M^{-3}\rho_r ,$$
$$\frac{[n']}{n}\bigg|_{y=y_r} = \tfrac{1}{3}M^{-3}(2\rho_r + 3p_r) , \quad (71)$$

where $[a']_{y=y_r} = a'(y = y_r^+) - a'(y = y_r^-) = -2a'(y_r^-)$. Since $a(y)$ is fixed by (6) or (13), the energy density ρ_r on the regulator brane can not be chosen independently, but is in fact directly related to the energy density ρ_b on the physical brane.

The condition (71) gives an additional term in the potential for the wavefunction,

$$V(z) = \frac{15\mu^2}{4(1+\mu|z|)^2} - 3\mu\delta(z) + \frac{3\mu}{1+\mu z_r}\delta(z - z_r) , \quad (72)$$

and thus the new boundary condition

$$2u'(z_r) - 3A'(y_r)u(z_r) = 2u'(z_r) + \frac{3\mu}{1+\mu z_r}u(z_r) = 0 , \quad (73)$$

in addition to (52). The zero mode (55) is trivially seen to satisfy this new condition. But the continuous spectrum of massive modes is reduced to a discreet spectrum. Using (54) together with well known identities for the Bessel functions, (73) can be simplified to

$$\frac{Y_1(\frac{m}{\mu})}{J_1(\frac{m}{\mu})} = \frac{Y_1[\frac{m}{\mu}(1+\mu z_r)]}{J_1[\frac{m}{\mu}(1+\mu z_r)]} \stackrel{z_r\to\infty}{\longrightarrow} \tan(m z_r - \tfrac{3\pi}{4}) , \quad (74)$$

which means that the mass is approximately quantized in units of π/z_r:

$$m_n \simeq \frac{n\pi}{z_r}\,, \quad n=0,1,2,3,\ldots \tag{75}$$

The transition from a sum over m to a continuous integral in the limit $z_r \to \infty$ is therefore given by

$$\sum_m f(m) = \sum_m f(m)\frac{z_r}{\pi}\Delta m \to \int_0^\infty f(m)\frac{z_r}{\pi}dm\,. \tag{76}$$

Next, we consider the normalization constant N_m when using a regulator brane, which is found from

$$\int_{-z_r}^{z_r} u_i(z)u_j(z)dz = 2\int_0^{z_r} u_i(z)u_j(z)dz = \delta_{ij}\,, \tag{77}$$

since the eigenfunctions $u_i(z) \equiv u(m_i,z)$ in (54) are real. When $i \neq j$, we can use (48) and write the integrand as

$$u_i(z)u_j(z) = \frac{1}{m_i^2 - m_j^2}\left[u_i(z)u_j''(z) - u_j(z)u_i''(z)\right]. \tag{78}$$

Performing a partial integration and using the boundary conditions at both endpoints, the integral then vanishes identically in the case $i \neq j$. For $i = j$ we get

$$\begin{aligned}\frac{1}{2} = \int_0^{z_r} u^2(m,z)dz = \int_0^{z_r} N_m^2(1+\mu z) \\ \times\left\{Y_2[\tfrac{m}{\mu}(1+\mu z)] - \frac{Y_1(\frac{m}{\mu})}{J_1(\frac{m}{\mu})}J_2[\tfrac{m}{\mu}(1+\mu z)]\right\}^2 dz\,. \end{aligned}\tag{79}$$

In the limit $z_r \to \infty$ we can use the asymptotic expressions for Y_2 and J_2 for large arguments, with the result

$$\begin{aligned}(z_r N_m^2)^{-1} &\to \lim_{z_r\to\infty}\frac{1}{z_r}\int_0^{z_r}\frac{4\mu}{\pi m} \\ &\quad\times\left\{\sin(mz - \tfrac{5\pi}{4}) - \frac{Y_1(\frac{m}{\mu})}{J_1(\frac{m}{\mu})}\cos(mz - \tfrac{5\pi}{4})\right\}^2 dz \\ &= \frac{2\mu}{\pi m}\left[1 + \frac{Y_1^2(\frac{m}{\mu})}{J_1^2(\frac{m}{\mu})}\right], \end{aligned}\tag{80}$$

or

$$N_m^2 = \frac{\pi m}{2\mu z_r}\left[1 + \frac{Y_1^2(\frac{m}{\mu})}{J_1^2(\frac{m}{\mu})}\right]^{-1}. \tag{81}$$

For the zero mode (55) we take the limit $z_r \to \infty$ directly, with the result

$$N_0^2 = \left[\int_{-\infty}^\infty \frac{dz}{(1+\mu|z|)^3}\right]^{-1} = \mu\,. \tag{82}$$

Inserting everything into (69), we see that the factor z_r from the integration measure and from the normalization constant cancel, as it should. We then finally obtain the gravitational potential

$$\begin{aligned}V(r) &= \frac{Gm_1m_2}{\mu r}\left\{u^2(0,0) + \frac{4}{3}\int_0^\infty u^2(m,0)e^{-mr}\frac{z_r}{\pi}dm\right\} \\ &= \frac{Gm_1m_2}{r}\left\{1 + \frac{2}{3\mu^2}\int_0^\infty me^{-mr}\right. \\ &\qquad\left.\times\frac{\left[J_1(\frac{m}{\mu})Y_2(\frac{m}{\mu}) - Y_1(\frac{m}{\mu})J_2(\frac{m}{\mu})\right]^2}{J_1^2(\frac{m}{\mu}) + Y_1^2(\frac{m}{\mu})}dm\right\} \\ &= \frac{Gm_1m_2}{r}\left\{1 + \frac{8}{3\pi^2}\int_0^\infty \frac{e^{-mr}}{J_1^2(\frac{m}{\mu}) + Y_1^2(\frac{m}{\mu})}\frac{dm}{m}\right\}, \end{aligned}\tag{83}$$

where we have used the general identity $J_n(x)Y_{n+1}(x) - Y_n(x)J_{n+1}(x) = -2/\pi x$ in the last line. The first term clearly gives 4-dimensional gravity, and the integral over the massive modes gives the correction due to the fifth dimension. Since the only parameter in the integral is μr (after substituting the dimensionless variable $n = m/\mu$), we see that the scale where the correction becomes important is given by the curvature radius μ^{-1} of the fifth dimension. For distances r much larger than this, the exponential factor ensures that 4-dimensional gravity is restored. The result (83) is the same as that obtained by Chung *et al.* [8].

The integral in (83) must in general be done numerically, which is straightforward using Mathematica, and the result is shown in figure 1. It is also possible to expand the result in powers of μr for the two regions $\mu r \ll 1$ and $\mu r \gg 1$. Writing the full potential as $V(r) = V_0(r)(1+\Delta)$, the relative correction Δ has the series expansion

$$\Delta = \begin{cases} \frac{4}{3\pi\mu r} - \frac{1}{3} - \frac{1}{2\pi}\mu r\ln\mu r \\ \qquad + 0.089237810\mu r + \mathcal{O}(\mu^2r^2)\,, & \mu r \ll 1, \\ \frac{2}{3\mu^2r^2} - \frac{4\ln\mu r}{\mu^4r^4} + \frac{16 - 12\ln 2}{3\mu^4r^4} \\ \qquad + \mathcal{O}\left[\frac{(\ln\mu r)^2}{\mu^6r^6}\right], & \mu r \gg 1. \end{cases}\tag{84}$$

A detailed derivation of this result is included in appendix A. Note in particular the leading order term $\Delta \simeq 4/(3\pi\mu r)$ for short distances. When $\mu r \ll 1$, $\Delta \gg 1$ and the entire gravitational potential behaves like $V \sim 1/r^2$. Gravity is therefore 5-dimensional at short distances. This should not come as a surprise, since when the distance r is small compared to the curvature radius μ^{-1} of the fifth dimension, spacetime looks almost flat and we should get the same result as with 5-dimensional Minkowski space.

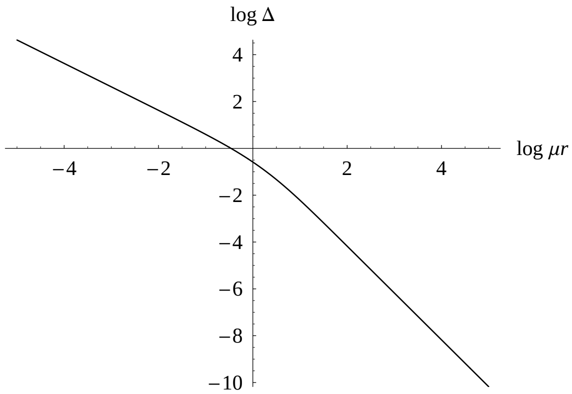


FIG. 1: The figure shows the relative correction Δ in (83), where $V(r) = V_0(r)(1+\Delta)$, as obtained by a numerical integration using Mathematica. The leading order terms $\Delta \simeq 4/(3\pi\mu r)$ (short distances) and $\Delta \simeq 2/(3\mu^2 r^2)$ (large distances) are clearly seen from the figure, and the transition between the two regions occurs around $\mu r \sim 1$.

V. SUMMARY

In this paper, we have gone through the derivation of the Newtonian gravitational potential in the RS model, from start to finish. We have concentrated on the critical case ($\Lambda = 0$ and $k = 0$), and have obtained the result (for large distances) $V \sim \frac{1}{r}(1 + \frac{2}{3\mu^2 r^2} + \ldots)$ from the graviton propagator, by including its full tensor structure. This is exactly the same result as what is found from a matter source on the brane when the brane-bending effect is included. As for the non-critical case, since both Λ and k/a_0^2 are very small (from observational constraints), corrections due to Λ and k will be noticable only at cosmological distances, and the correction due to the fifth dimension must therefore be the same as for the critical case (83).

Throughout this paper, we have assumed that the dark radiation term vanishes ($U = 0$), and we have ignored contributions from matter and radiation on the brane by setting $\rho = \rho_m + \rho_r = 0$. In a realistic model, however, these terms have to be included. Because of the degeneracy between dark radiation and ordinary radiation, the two of them can not be distinguished by studying cosmological evolution alone. But since no observation has been made that requires the presence of dark radiation, we can probable assume that it is small, i.e. that $|U| \lesssim \rho_r a_0^4/M_{\rm Pl}^2$.

It is interesting to note, however, that the scale factor $a(t,y)$ may change dramatically far from the brane when including even a tiny value of ρ or U, which is seen from the following example: For $\Lambda_B < 0$ and $\Lambda = 0$ we assume that all parameters of the model (M, μ and λ) are of the order of the Planck scale, which means that we ignore the ρ^2 term in (10). Combining this with (6), the scale factor reduces to

$$\frac{a^2(t,y)}{a_0^2(t)} = e^{-2\mu|y|} - \frac{\rho\left(1 - e^{-2\mu|y|}\right)}{6\mu^2 M_{\rm Pl}^2} - \frac{U \sinh^2 \mu y}{\mu^2 a_0^4}\,. \quad (85)$$

Thus, both ρ and U introduce a horizon at a finite distance from the brane, unless U is negative, in which case the scale factor reaches a minimum and then diverges as $y \to \infty$. The same two effects arise when $\Lambda > 0$ and $\Lambda < 0$, respectively. However, as shown by Karch and Randall [14], gravity can still be localized in these cases, since localization only depends on the scale factor close to the brane. We should therefore expect that matter and (dark) radiation only play a significant role in the very early universe, when ρ and U/a_0^4 are of the order of the Planck scale.

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APPENDIX A: SERIES EXPANSION OF Δ

We are considering the integral

$$\Delta = \frac{8}{3\pi^2}\int_0^\infty \frac{e^{-\mu r n}}{J_1^2(n) + Y_1^2(n)}\frac{dn}{n} \equiv \int_0^\infty f(n) e^{-\mu r n} dn\,, \quad (A1)$$

where we are using the dimensionless integration variable $n = m/\mu$. In the following, we will need the series expansion of $f(n)$ for small and large values of n, which is found using Mathematica:

$$\begin{aligned} f(n \ll 1) = {}& \frac{2}{3}n + \frac{2}{3}\left(-\frac{1}{2} + \gamma - \ln 2 + \ln n\right) n^3 \\ & + \left[-0.107720074417 + \frac{1}{2}(\ln n)^2 \right. \\ & \left. + \left(\gamma - \ln 2 - \frac{7}{12}\right)\ln n\right] n^5 + \ldots, \quad (A2) \end{aligned}$$

$$f(n \gg 1) = \frac{4}{3\pi} - \frac{1}{2\pi n^2} + \frac{21}{32\pi n^4} - \frac{633}{256\pi n^6} + \ldots \quad (A3)$$

1. Short distances, $\mu r \ll 1$

In the limit of short distances, the exponential factor $e^{-\mu r n}$ becomes important when n is large and we can use (A3). However, this series can not be integrated term by term from 0 to ∞, since all terms (except the first one) diverges in the limit $n \to 0$. We therefore split the integral into two parts, $\int_0^\infty = \int_0^N + \int_N^\infty$, where N is assumed to be large. In the first part we expand $e^{-\mu r n}$, and the second part is integrated term by term, using the general formula

$$\begin{aligned} \int_N^\infty \frac{e^{-\mu r n}}{n^k} dn = {}& e^{-\mu r N}\sum_{i=0}^{k-2} \frac{(k-2-i)!}{(k-1)!}\frac{(-\mu r)^i}{N^{k-1-i}} \\ & + \frac{(-\mu r)^{k-1}}{(k-1)!}\Gamma(0, \mu r N)\,. \quad (A4) \end{aligned}$$

Here $\Gamma(a,z)$ is the incomplete gamma function $\Gamma(a,z) = \int_z^\infty t^{a-1}e^{-t}dt$. Writing the expansion of $f(n)$ as $f(n) = \sum_{k=0}^\infty C_{2k}n^{-2k}$, we then obtain

$$\Delta = \sum_{k=0}^{\infty} \frac{(-\mu r)^k}{k!} \int_0^N n^k f(n) dn + \frac{4e^{-\mu r N}}{3\pi\mu r} + \sum_{k=1}^{\infty} C_{2k} \left[e^{-\mu r N} \sum_{i=0}^{2k-2} \frac{(2k-2-i)!}{(2k-1)!} \frac{(-\mu r)^i}{N^{2k-1-i}} - \frac{(\mu r)^{2k-1}}{(2k-1)!}\Gamma(0,\mu r N) \right] . \tag{A5}$$

This expression can now be expanded in powers of μr, using

$$\Gamma(0,\mu r N) = -\gamma - \ln\mu r - \ln N - \sum_{k=1}^{\infty} \frac{(-\mu r N)^k}{k!\cdot k}\,, \tag{A6}$$

and we can then extract and calculate the coefficient in front of each power of μr in turn. From (A6), we see that we also get several logarithmic terms. For each power of μr, we take the limit $N\to\infty$, therefore ignoring all terms with a negative power of N. Using the notation

$$\Delta = \sum_{k=-1}^{\infty} A_k (\mu r)^k + \text{logarithmic terms}, \tag{A7}$$

we first find rather trivially that $A_{-1} = 4/(3\pi)$, and then, using numerical integration:

$$\begin{aligned}
A_0 &= \lim_{N\to\infty} \left[\int_0^N f(n)dn - \frac{4N}{3\pi} \right] = \int_0^\infty \left[f(n) - \frac{4}{3\pi} \right] dn = -\frac{1}{3}\,, \\
A_1 &= \lim_{N\to\infty} \left[-\int_0^N nf(n)dn + \frac{2N^2}{3\pi} - \frac{1}{2\pi}\ln N \right] + \frac{1-\gamma}{2\pi} \\
&= -\int_0^1 n\left[f(n) - \frac{4}{3\pi} \right] dn - \int_1^\infty n \left[f(n) - \frac{4}{3\pi} + \frac{1}{2\pi n^2} \right] dn + \frac{1-\gamma}{2\pi} = 0.08923780957038536\,, \\
A_2 &= \lim_{N\to\infty} \left[\frac{1}{2}\int_0^N n^2 f(n)dn - \frac{2N^3}{9\pi} + \frac{N}{4\pi} \right] = \frac{1}{2}\int_0^\infty n^2 \left[f(n) - \frac{4}{3\pi} + \frac{1}{2\pi n^2} \right] dn = \frac{1}{8}\,, \\
A_3 &= \lim_{N\to\infty} \left[-\frac{1}{3!}\int_0^N n^3 f(n)dn + \frac{N^4}{18\pi} - \frac{N^2}{24\pi} + \frac{7}{64\pi}\ln N \right] + \frac{42\gamma - 77}{384\pi} \\
&= -\frac{1}{6}\int_0^1 n^3 \left[f(n) - \frac{4}{3\pi} + \frac{1}{2\pi n^2} \right] dn - \frac{1}{6}\int_1^\infty n^3 \left[f(n) - \frac{4}{3\pi} + \frac{1}{2\pi n^2} - \frac{21}{32\pi n^4} \right] dn + \frac{42\gamma-77}{384\pi} \\
&= -0.02955870986828890\,, \\
A_4 &= \lim_{N\to\infty} \left[\frac{1}{4!}\int_0^N n^4 f(n)dn - \frac{N^5}{90\pi} + \frac{N^3}{144\pi} - \frac{7N}{256\pi} \right] \\
&= \frac{1}{24}\int_0^\infty n^4 \left[f(n) - \frac{4}{3\pi} + \frac{1}{2\pi n^2} - \frac{21}{32\pi n^4} \right] dn = -\frac{3}{128}\,.
\end{aligned} \tag{A8}$$

Note in particular how the first terms in the expansion of $f(n)$ act as counterterms, making all the integrals converge, and the numerical calculation therefore straightforward using Mathematica. The coefficient in front of the logarithmic terms is easily found from (A5) and (A6):

$$\sum_{k=1}^{\infty} \frac{C_{2k}}{(2k-1)!}(\mu r)^{2k-1} = -\frac{1}{2\pi}\mu r + \frac{7}{64\pi}(\mu r)^3 + \ldots \tag{A9}$$

The complete series expansion of Δ is therefore obtained as

$$\Delta = \frac{4}{3\pi\mu r} - \frac{1}{3} + 0.089237810\mu r - \frac{1}{2\pi}\mu r\ln\mu r + \frac{1}{8}\mu^2 r^2 - 0.029558710\mu^3 r^3 + \frac{7}{64\pi}\mu^3 r^3 \ln\mu r - \frac{3}{128}\mu^4 r^4 + \mathcal{O}(\mu^5 r^5)\,. \tag{A10}$$

2. Large distances, $\mu r \gg 1$

In the limit of large distances, the series expansion of Δ is obtained a lot easier than in the previous section, since we can now integrate the expansion of $f(n)$ in (A2) for small n directly from 0 to ∞. The result is therefore

$$\Delta = \frac{2}{3\mu^2 r^2} + \frac{16 - 12\ln 2}{3\mu^4 r^4} - \frac{4\ln\mu r}{\mu^4 r^4} + \frac{29.4398446730}{\mu^6 r^6} + \frac{(120\ln 2 - 204)\ln\mu r}{\mu^6 r^6} + \frac{60(\ln\mu r)^2}{\mu^6 r^6} + \mathcal{O}(1/\mu^8 r^8)\,. \tag{A11}$$

The error that follows from using the expansion (A2) all the way to $n = \infty$ is of exactly the same order as what we would get from the next term in the expansion, which is easily shown: If we denote the first terms in the expansion by $h(n)$, such that $f(n) = h(n) + \mathcal{O}(n^7)$ for small n, the relative correction to the gravitational potential can be written $\Delta = \int_0^\infty f(n)e^{-\mu r n}dn = \int_0^\infty h(n)e^{-\mu r n}dn + \delta$, where the error δ is bounded by

$$|\delta| \le \int_0^\infty e^{-\mu r n}\left|f(n) - h(n)\right| dn\,. \tag{A12}$$

It turns out that $f(n) - h(n) = \frac{1}{3}n^7(\ln n)^3 +$ higher order terms. A rough estimate $E(n)$ of the difference $f(n) - h(n)$ which satisfies $E(n) > |f(n) - h(n)|$ for all n can therefore be chosen as $E(n) = n^7\left[1 - (\ln n)^3 e^{-n}\right]$. This means that

$$|\delta| < \int_0^\infty e^{-\mu r n}E(n)dn \sim \frac{(\ln\mu r)^3}{\mu^8 r^8} + \text{higher order terms}, \tag{A13}$$

i.e. precisely the same leading order term as what is obtained from the first term not included in (A2).

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